

# The Markowitz Category

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## Abstract

We give an algebraic definition of a Markowitz market and classify markets up to isomorphism. Given this classification, the theory of portfolio optimization in Markowitz markets without short selling constraints becomes trivial. Conversely, this classification shows that, up to isomorphism, there is little that can be said about a Markowitz market that is not already detected by the theory of portfolio optimization.

## Introduction

This note is motivated by the celebrated one and two mutual fund theorems [3]. These build on the work of Markowitz in [2] and tell us that, in the Markowitz market model with no restrictions on short selling, an investor who is only interested in the optimal investment problems can safely ignore all but a two-dimensional subspace of the space of portfolios. But what if one's interests are more broad ranging than these optimal investment problems? Are there other low-dimensional subspaces of the market model that may be of particular interest to other market players?

We will prove that, in some sense, the answer to this question is no. Moreover, the two mutual fund theorem in some sense says all that there is to say about the market.

The key, of course, is to explain what we mean by “in some sense”. This is where we use a little category theory.

Category theory, introduced in [1], formalises the common practice of mathematicians to investigate categories of object up to some notion of equivalence or isomorphism. For example, one might attempt to classify vector spaces up to bijective linear transformation or finite groups up to group isomorphism. The advantage of this approach is that spurious details are ignored. For example the specific set underlying the vector space or the group are irrelevant to their classification up to isomorphism.

Following a similar pattern, in Section 1 we will define a class of objects called Markowitz markets and define a notion of a Markowitz isomorphism between markets. By defining the notion of isomorphism we formally define what we consider to be a financially meaningful feature of a Markowitz market, and what

we consider to be spurious information. For example the name of a specific stock is not financially meaningful and our notion of isomorphism should reflect this.

Having identified the notion of isomorphism, we will then classify all arbitrage free Markowitz markets up to Markowitz isomorphism. This requires only elementary linear algebra and can be done without considering portfolio optimization at all.

In Section 2 we will show how our classification of Markowitz markets can be applied to the study of portfolio optimization. We will see that classical results such as the mutual fund theorems are immediately obvious corollaries of our classification. Moreover, we will see the close relation between risk-return diagrams and the classification of markets. For example, we will see that two markets of the same dimension and containing no spurious portfolios of zero cost, zero risk and zero expected payoff are isomorphic if and only if they have the same efficient frontier.

In Section 3 we formally state and prove a mathematical version of our claim that there are no low-dimensional subspaces of the market model that may be of particular interest to other market players. Our essential assumption in proving this result is that market players are only interested in markets up to Markowitz isomorphism. For example, if they are interested in risk measures other than just standard deviation, this assumption would be violated. Note however, that we do not make any distributional assumptions about the returns in our market other than that the first two moments exist. In particular, the normal distribution plays no role in our theory.

Our assumption of invariance up to Markowitz isomorphism is also violated by a market player who knows that they need to hedge a particular portfolio. This financial problem is considered by [4]. Such a market player should use a different category of objects consisting of pairs of a market and a portfolio to hedge. Their notion of morphism would be restricted to morphisms that preserve the hedging portfolio. This is a special case of a “pointed category”. We give a classification of the pointed category of markets and a marked portfolio in section 2, and our entire approach can be generalized easily to cover the financial problem of [4].

We remark that in the widespread practice of applying principal component analysis to the covariance matrix, one is also breaking invariance up to Markowitz isomorphism. The reason is that to perform a principal component analysis one must first fix a reference Euclidean metric on the space of portfolios and this must be chosen independently of the covariance form. This Euclidean metric can be used to convert the covariance form into a covariance matrix and hence compute eigenvectors and eigenvalues of the covariance matrix. Before performing a principal component analysis, one must identify this Euclidean metric and justify its financial relevance. Our point is that if one is only interested in markets up to Markowitz isomorphism, no financially relevant Euclidean metric exists and principal component analysis is meaningless.

Our category theory approach to portfolio optimization is in many ways more general and more illuminating than the classical approach of [3] (and which has been repeated ad-infinitum in textbooks and student dissertations).

We use geometric arguments based around the Gram–Schmidt process, whereas the classical approach is based on direct calculation and the theory of Lagrange multipliers. This geometric approach to portfolio theory has surely been observed before, yet it seems to have had little impact on the textbooks. We hope that our explicit classification theory of markets might popularise the geometric approach.

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## 1 The Markowitz Category

**Definition 1.1.** A *Markowitz market*  $(V, r, c, p)$  consists of a finite dimensional real vector space  $V$  together with the data:

- (i) A symmetric bilinear map  $r : V \times V \rightarrow \mathbb{R}$  satisfying  $r(v, v) \geq 0$  for all  $v \in V$ ;
- (ii) Two linear functionals  $c : V \rightarrow \mathbb{R}$  and  $p : V \rightarrow \mathbb{R}$ .

**Definition 1.2.** A *Markowitz morphism* between two Markowitz markets  $(V, r, c, p)$  and  $(V', r', c', p')$  is a linear transformation  $T : V \rightarrow V'$  which satisfies:

$$r'(Tv, Tv) = r(v, v) \quad \forall v \in V, \quad (1)$$

$$p'(Tv) = p(v) \quad \forall v \in V, \quad (2)$$

$$c'(Tv) = c(v) \quad \forall v \in V. \quad (3)$$

Two Markowitz markets are said to be isomorphic if there is a bijective Markowitz morphism from one to the other.

Markowitz markets naturally arise in finance.

Consider a trader who buys and sells  $n$  financial assets. The trader is interested in studying portfolios made up from these assets. A portfolio is defined by knowing the vector in  $\mathbb{R}^n$  that contains the quantity of each asset held. The abstract vector space  $V$  in our definition of a Markowitz market represents the space of possible portfolios. A portfolio may contain a negative quantity of a particular asset, this is interpreted financially by saying that a trader may choose to buy assets (a positive quantity) or borrow them (a negative quantity).

In this financial setting, the linear functional  $c$  computes the initial cost of setting up a portfolio. If we assume the market is infinitely liquid and that unlimited amounts of each asset can be bought and sold it is reasonable to assume that the cost is indeed linear.

The trader models the financial assets as random variables. The linear functional  $p$  computes the expected payoff of the portfolio at some future time  $T$ .

Infinite liquidity and infinite market depth justify the assumption that  $p$  is linear. The symmetric bilinear map  $r$  computes the covariance of the two portfolios at the future time  $T$ . Note that here we are assuming that all the assets have finite variance.

The quantity  $\sqrt{r(v, v)}$ , (the standard deviation of  $v$ ), should be thought of as the *risk* of a portfolio  $v$ . There is an extensive literature on risk measurement and numerous statistical quantities have been proposed that can be used to measure the risk of a portfolio. We will not debate the pros and cons of different risk measures here, we simply state that, in the Markowitz framework, risk is measured using standard deviation.

To justify the definition of a Markowitz morphism we assume that the trader is only interested in the portfolios that are available, their costs, payoffs and risk measured using the standard deviation. The trader sees all other market data as extraneous. In particular the trader is unconcerned by the question of how many assets are combined to produce a portfolio.

Our aim now is to classify Markowitz markets up to isomorphism. This is an elementary exercise in linear algebra. To reduce the number of cases in our classification, we will only classify arbitrage-free markets. These are defined as follows.

**Definition 1.3.** An *arbitrage portfolio* is a portfolio  $v \in V$  satisfying  $r(v) = 0$ ,  $c(v) = 0$  and  $p(v) > 0$ . A Markowitz market is *arbitrage free* if it does not contain any arbitrage portfolios.

**Definition 1.4.** A portfolio  $v \in V$  is said to be *risk free* if  $r(v, v) = 0$ . A portfolio  $v \in V$  is said to be *costless* if  $c(v) = 0$ . A portfolio  $v \in V$  is said to be *valueless* if  $r(v, v) = 0$ ,  $c(v) = 0$  and  $p(v) = 0$ .

**Lemma 1.5.** If  $T$  is a Markowitz morphism between  $(V, r, c, p)$  and  $(V', r', c', p')$  then

$$r'(Tv_1, Tv_2) = r(v_1, v_2) \quad \forall v_1, v_2 \in V.$$

*Proof.* This follows immediately from the polarization identity for symmetric bilinear maps:

$$r(v_1, v_2) = \frac{1}{4} (r(v_1 + v_2, v_1 + v_2) - r(v_1 - v_2, v_1 - v_2)). \quad (4)$$

This shows that the entire covariance structure  $r$  can be deduced from knowing the standard deviation  $r(v, v)$ .  $\square$

**Lemma 1.6.** Define the linear map  $\tilde{r} : V \rightarrow V^*$  by  $\tilde{r}(v)(w) = r(v, w)$  then the set of risk free portfolios,  $V^0$ , is equal to  $\ker \tilde{r}$ .

*Proof.* If  $v \in \ker \tilde{r}$  then  $r(v, v) = \tilde{r}(v)(v) = 0$ . So  $\ker \tilde{r} \subseteq V^0$ .

On the other hand, if  $r(v, v) = 0$  then the function  $n(v) = r(v, v)$  has a local minimum at  $v$ . So the derivative of  $n$  in any direction  $w \in V$  is equal to zero. This derivative is equal to  $2r(v, w) = 2\tilde{r}(v)(w)$ . So  $V^0 \subseteq \ker \tilde{r}$ .  $\square$

**Corollary 1.7.** *If we have a decomposition  $V = V^0 \oplus V^1$  for some vector subspace  $V^1$  then the value of  $r$  on  $V$  is determined by its value on  $V^1$ .*

*Proof.* Let  $v = v_0 + v_1$  where  $v_0 \in V^0$  and  $v_1 \in V^1$ . Then

$$\begin{aligned} r(v, v) &= r(v_0, v_0) + 2r(v_0, v_1) + r(v_1, v_1) \\ &= r(v_1, v_1) \end{aligned}$$

The result now follows from Lemma 1.5.  $\square$

If a portfolio satisfies  $r(v) = 0$ ,  $c(v) = 0$  and  $p(v) \neq 0$  then either  $v$  or  $-v$  will be an arbitrage portfolio. So a Markowitz market is arbitrage free if and only if all costless, risk-free portfolios are valueless. This yields the following result:

**Lemma 1.8** (Classification of arbitrage-free riskless markets). *In an arbitrage-free Markowitz market, we can write  $V^0 = V^R \oplus ((\ker c) \cap V^0)$  where  $V^R$  is zero or one dimensional. If  $V^R$  is one dimensional it is spanned by a single portfolio  $v_R$  of cost 1.  $p = 0$  on  $(\ker c) \cap V^0$ .*

We are now ready to state and prove our main result which is to give a canonical form for all arbitrage free Markowitz markets.

The canonical forms will be expressed in terms of the vector space  $\mathbb{R}^n$ . We will write the bilinear map  $r$  on  $\mathbb{R}^n$  as an  $n \times n$  matrix  $\mathbf{r}$  such that

$$r(v, w) = v^T \mathbf{r} w.$$

We will write the linear functionals  $c$  and  $p$  as co-vectors. We will write the matrices  $\mathbf{r}$  in block diagonal form and will use the notation  $1_k$  for the  $k \times k$  identity matrix and will use  $0$  for matrices of zeros whose dimensions can be deduced from the context.

**Theorem 1.9.** *We have the following classification of Markowitz markets.*

(a) The case  $c \neq 0$ .

*Let  $n$  be given. Given four parameters  $(k, m, g, i) \in \{0, 1, \dots, n\} \times [0, \infty) \times [0, \infty) \times \mathbb{R}$  which do not lie in the set*

$$E_n = \{(k, m, g, i) : (k = n \text{ and } m = 0) \text{ or } (k = 0 \text{ and } m \neq 0)\} \quad (5)$$

*we can define an isomorphism class of Markowitz markets,  $\mathcal{M}_{k,m,g,i}^n$ , as follows:*

(i) *If  $m = 0$ ,  $\mathcal{M}_{k,m,g,i}^n$  is the isomorphism class of the market  $\mathbb{R}^n$  with*

$$\mathbf{r} = \begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}, \quad c = (0, 0, \dots, 0, 1), \quad p = (g, 0, \dots, 0, i).$$

(ii) If  $m \in (0, \infty)$ ,  $\mathcal{M}_{k,m,g,i}^n$  is the isomorphism class of the market  $\mathbb{R}^n$  with

$$\mathbf{r} = \begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}, \quad c = \left( \frac{1}{m}, 0, \dots, 0 \right), \quad p = \begin{cases} \left( \frac{i}{m}, 0, \dots, 0 \right) & \text{if } k = 1 \\ \left( \frac{i}{m}, g, 0, \dots, 0 \right) & \text{otherwise.} \end{cases}$$

Note that when  $k = 1$  the parameter  $g$  is ignored. We have chosen our coordinates  $m$  and  $i$  for the isomorphism classes so that these variables will have simple geometric and financial explanations. This justifies the apparently unnecessary complexity of using  $\frac{1}{m}$  and  $\frac{i}{m}$  in the formulae.

Any arbitrage-free Markowitz market of dimension  $n$  with  $c \neq 0$  belongs to one of these isomorphism classes. The isomorphism classes  $\mathcal{M}_{k,m,g,i}^n$  are distinct except that

$$\text{if } m \in (0, \infty), \text{ then } \mathcal{M}_{1,m,g,i}^n = \mathcal{M}_{1,m,g',i}^n \quad \forall g, g'. \quad (6)$$

(b) The case  $c = 0$ .

Any arbitrage-free Markowitz market of dimension  $n$  with  $c$  identically zero is Markowitz isomorphic to the market  $\mathbb{R}^n$  with

$$\mathbf{r} = \begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}, \quad c = (0, 0, \dots, 0), \quad p = (g, 0, \dots, 0)$$

where  $k$  is a uniquely determined integer between 0 and  $n$ .  $g = 0$  if  $k = 0$  but otherwise,  $g$  is a uniquely determined element of  $[0, \infty)$ .

*Proof.* We first assume that  $c \neq 0$ . Case (i) and (ii) can be distinguished in an invariant fashion since there is a risk free portfolio  $v_R$  with  $c(v_R) \neq 0$  in case (i) but not in case (ii). Let us show that conversely if there is such a portfolio we can find a basis such that the market takes the form of case (i), and if not, it takes the form in case (ii).

(i) We suppose that a risk free portfolio with non zero cost  $v_R$  exists. Take  $e_n = \frac{v_R}{c(v_R)}$  and take  $k = n - \dim V^0$ . Take  $\{e_{k+1}, \dots, e_{n-1}\}$  to be a basis for  $(\ker c) \cap V^0$ . By Lemma 1.8,  $p$  is equal to 0 on  $(\ker c) \cap V^0$ . Extend  $\{e_{k+1}, \dots, e_{n-1}\}$  to a basis  $\{v_1, \dots, v_k, e_{k+1}, \dots, e_{n-1}\}$  for  $\ker c$ . Let  $V_k$  be the span of  $\{v_1, \dots, v_k\}$ . Then  $r$  restricted to  $V_k$  gives an inner product, so by applying the Gram-Schmidt process we can find an orthonormal basis  $\{e_1, \dots, e_k\}$  for  $r$  restricted to  $V_k$ . The inner product on  $V_k$  gives a duality isomorphism from  $V_k$  to  $V_k^*$ . Let  $v_p$  denote the vector in  $V_k$  that is dual to the functional  $p \upharpoonright_{V_k}$  via this isomorphism. By applying an isometry of the Euclidean space  $V_k$  if necessary, we may assume that  $v_p$  is a non-negative multiple of  $e_1$ . When one writes  $r$ ,  $c$  and  $p$  with respect to the basis  $\{e_1, \dots, e_n\}$  we see from Corollary 1.7 that they take the desired form.

Given that the market is of this form,  $i$  can be invariantly defined as the expected payoff of a riskless portfolio of cost 1. In the same circumstances,  $g$  can be invariantly defined as the maximum value of  $p$  among costless portfolios  $v$  with  $r(v, v) \leq 1$ . It follows that  $i$  and  $g$  are uniquely determined.

- (ii) We suppose that all risk free portfolios have cost zero. Take  $k = n - \dim V^0$ . Take  $\{e_{k+1}, \dots, e_n\}$  to be a basis for  $V^0$ . Extend this to get a basis  $\{v_1, \dots, v_k, e_{k+1}, \dots, e_n\}$  for  $V$ . Let  $V_k$  denote the span of the  $v_k$ . It is an inner product space with respect to  $r$ , so by applying the Gram-Schmidt process we can obtain a basis  $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$  for  $V$  with the  $\{e_1, \dots, e_k\}$  orthonormal. By applying an isometry of  $V_k$  if necessary, we may assume that the vector dual to  $c$  via the inner product on  $V_k$  is a positive multiple of  $e_1$ . By applying a further isometry of the space spanned by  $e_2, \dots, e_k$ , we may assume that the vector dual to  $p$  via the inner product on  $V_k$  lies in the span of  $e_1$  and  $e_2$ . Writing the market with respect to this basis now puts it into the desired form.

Given that the market is of this form,  $m$  can be defined invariantly as 1 over the maximum cost of any portfolio  $v$  with  $r(v, v) = 1$ . Define  $i'$  invariantly as the payoff  $p(v)$  of a portfolio with  $r(v, v) = 1$  that maximizes the cost. Now  $i$  can be defined invariantly by  $i' = \frac{i}{m}$ .  $g$  can be defined invariantly as the maximum expected payoff of any costless portfolio  $v$  with  $r(v, v) = 1$ .

The proof for the case when  $c = 0$  is similar.  $\square$

To avoid considering financially-uninteresting special cases in the sequel we make the following definition.

**Definition 1.10.** A Markowitz market is *non-degenerate* if:

- (i) The market is arbitrage free;
- (ii) There are no valueless portfolios;
- (iii)  $c$  and  $p$  are linearly independent.

It follows from our theorem that all non-degenerate Markowitz markets of dimension  $n$  are of the form  $\mathcal{M}_{n-1,0,g,i}$  or  $\mathcal{M}_{n,m,g,i}$  with  $m \in (0, \infty)$  and  $g \in (0, \infty)$ .

We have identified the set of non-degenerate Markowitz markets up to isomorphism. We now ask what is the topology of this space?

For a fixed underlying vector space,  $V$  we can choose an isomorphism to  $\mathbb{R}^n$ . The space of bilinear forms on  $V$  can then be viewed as a subspace of  $\mathbb{R}^{n^2}$  and so can be given a topology. We can then give the space of Markowitz markets on  $V$  a topology. This topology doesn't depend upon the choice of isomorphism from  $V$  to  $\mathbb{R}^n$ . Thus the space of Markowitz markets has a natural topology. The moduli space of Markowitz markets is defined to be the quotient of the space of Markowitz markets by the equivalence relation given by Markowitz isomorphisms.

With this terminology established we may now prove the following corollary of Theorem 1.9.

**Corollary 1.11.** *The moduli space of non-degenerate Markowitz markets of dimension  $n \geq 3$  is homeomorphic to the manifold with boundary  $[0, \infty) \times (0, \infty) \times \mathbb{R}$ . In particular, the map  $\tau$  given by  $\tau(m, g, i) = \mathcal{M}_{n-\delta_0(m), m, g, i}$  is a homeomorphism. Here  $\delta_0(m)$  is equal to 1 if  $m = 0$  and equal to 0 otherwise.*

*Proof.* It follows from Theorem 1.9 that  $\tau$  is a bijection.

Define  $\tilde{\tau}(m, g, i)$  to be the market given in matrix form by

$$\mathbf{r} = \begin{pmatrix} m^2 & 0 \\ 0 & I_{n-1} \end{pmatrix}, \quad c = (1, 0, 0, \dots, 0), \quad p = (i, g, 0, \dots, 0).$$

$\tilde{\tau}$  is continuous. The market  $\tilde{\tau}(m, g, i)$  is Markowitz isomorphic to  $\tau(m, g, i)$ . Therefore  $\tau$  is continuous.

We can invariantly and continuously associate a non-degenerate bilinear form  $\hat{r}$  with a non-degenerate Markowitz market by defining

$$\hat{r}(u, v) = r(u, v) + c(u)c(v).$$

To any non-degenerate bilinear form on a finite dimensional vector space, there is an associated isomorphism between the vector space and its dual. This isomorphism is associated continuously. Thus we can continuously and invariantly associate a bilinear form acting on  $V^*$  with any non-degenerate Markowitz market. We will write  $\hat{r}^*$  for this form.

A short calculation shows that in both cases (i) and (ii) of Theorem 1.9 we have  $\hat{r}^*(c, c) = \frac{1}{1+m^2}$ . Therefore

$$m = \sqrt{\frac{1}{\hat{r}^*(c, c)} - 1}.$$

Thus the function  $m$  defined on the moduli space of non-degenerate markets is continuous. We calculate similarly that  $\hat{r}^*(p, c) = \frac{i}{1+m^2}$  and  $\hat{r}^*(p, p) = \frac{i^2}{1+m^2} + g^2$ . Thus  $m$ ,  $i$  and  $g$  are continuous functions on the moduli space of non-degenerate Markowitz markets. Hence  $\tau^{-1}$  is continuous.  $\square$

## 2 Portfolio Optimization

Armed with our classification theorem, the study of portfolio optimization in Markowitz markets becomes entirely trivial.

**Definition 2.1.** Given a Markowitz market, a portfolio  $v_0$  is said to be *risk minimizing* if its risk  $r(v_0, v_0)$  is equal to the minimum risk among all portfolios,  $v$ , with  $c(v) = c(v_0)$  and  $p(v) = p(v_0)$ .



**Theorem 2.2** (Two mutual-fund theorem). *In a non-degenerate Markowitz market with no risk-free portfolios, the set of risk-minimizing portfolios is a vector subspace of  $V$  of dimension at most 2. Moreover, for any feasible payoff and cost there is an associated risk-minimizing portfolio. This is called the two mutual-fund theorem because the space of risk-minimizing portfolios can spanned by two portfolios, these are the “mutual-funds”.*

*Proof.* Since there are no non-zero risk-free portfolios, we are in case (ii) of our classification, Theorem 1.9. In this case, our vector space is Euclidean space with risk measured by distance, making the result geometrically obvious. We give a few formal details for completeness.

Two portfolios  $v$  and  $v_0$  have the same cost and expected payoff if and only if their first two components are equal. The risk is equal to the sum of the squares of the components, and hence is minimized by taking all components other than the first two equal to zero. Hence the space of risk-minimizing portfolios is the vector space spanned by the standard basis vectors  $\{e_1, e_2\}$ .  $\square$

**Theorem 2.3** (One mutual-fund theorem). *In a non-degenerate Markowitz market with a risk-free portfolio the set of risk-minimizing portfolios is a vector subspace of  $V$  of dimension at most 2 and contains the risk-free portfolio. For any feasible payoff and cost there is an associated risk-minimizing portfolio. This is called the one mutual-fund theorem because the space of risk-minimizing portfolios can spanned by one arbitrary portfolio and a risk-free portfolio.*

*Proof.* An obvious consequence of case (i) of Theorem 1.9  $\square$

We have not yet used the concept of *return* of a portfolio. In standard treatments of Markowitz’s theory it is usual to rescale investment problems in terms of the initial cost of a portfolio. This rescaling function is non-linear and not even defined for portfolios of zero cost. It often seems to unnecessarily complicate the discussion. For example, we have stated the mutual-fund theorems in terms of vector spaces which we believe makes them much easier to understand than conventional presentations.

However, the idea that one might be able to rescale and transform a market to simplify it is central to our discussion, it is simply that returns are the “wrong” rescaling. We have observed that the covariance structure  $r$  defines a natural length scale for the problem and have transformed our coordinates so that this becomes the standard Euclidean metric. This transformation has the advantage of being linear. This observation is generally useful throughout probability theory: covariance matrices define natural length scales.

**Definition 2.4.** The *expected return* of a portfolio,  $v$  with non-zero cost is given by

$$\text{ER}(v) := \frac{p(v) - c(v)}{c(v)}.$$

The *relative risk* of such a portfolio is given by

$$\text{RR}(v) := \sqrt{r(v, v)} / c(v).$$

Let  $\phi$  map the set  $V \setminus (\ker c)$  to  $\mathbb{R}^2$  by  $\phi(v) = (\text{RR}(v), \text{ER}(v))$ . The image of  $\phi$  is called the *feasible set*. The image of the set of risk-minimizing portfolios is called the *efficient frontier*. The shape of the efficient frontier was identified in [3].

**Theorem 2.5.** *In a non-degenerate Markowitz market,  $\mathcal{M}_{n-\delta_0(m),m,g,i}$  with  $n \geq 2$ , the efficient frontier consists of the points  $(x, y)$  with  $x \geq 0$  and*

$$g^2(x^2 - m^2) = (y + 1 - i)^2. \quad (7)$$

*When  $n = 2$ , the feasible set is equal to the efficient frontier. When  $n > 2$ , the feasible set is the set of all points on, or to the right of, the efficient frontier.*

*Proof.* We consider first case (ii) of Theorem 1.9 when  $m > 0$ . Because of the scaling by cost in the definition of ER and RR we see that we need only consider the image of portfolios of cost 1.

An efficient portfolio with cost 1 takes the form  $v = (m, \lambda, 0, \dots, 0)$  for some  $\lambda$ . It is mapped to:

$$\phi(v) = \left( \sqrt{m^2 + \lambda^2}, i + g\lambda - 1 \right).$$

We can compute  $g^2\lambda^2$  from either the  $x$ -coordinate or  $y$ -coordinate of  $\phi(v)$ . Equating these expressions gives the expression (7). Since  $g \neq 0$  we see that the  $y$ -coordinate of  $\phi(v)$  can take any real value, so the efficient frontier is the right arm of the hyperbola satisfying (7).

If  $n = 2$  all portfolios are efficient. If  $n > 2$ , the portfolio  $(m, \lambda, \mu, 0, \dots, 0)$  is mapped by  $\phi$  to

$$\left( \sqrt{m^2 + \lambda^2 + \mu^2}, i + g\lambda - 1 \right).$$

So any point to the right of the efficient frontier is feasible.

The efficient frontier and feasible set are similarly easy to calculate in case (i) of Theorem 1.9.  $\square$

The feasible set and the efficient frontier are iconic images of Markowitz's theory. They are illustrated in Figure 1.

We now see the justification for our choice of parameter names for the space of Markowitz markets. The parameter  $m$  measures the minimum risk of a portfolio of cost 1, the parameter  $g$  measures the gradient of the asymptotes when  $m > 0$  or the slope of the lines that the hyperbola degenerates to when. The parameter  $i - 1$  corresponds to the  $y$ -axis intercept on the  $y$ -axis where the asymptotes meet.

From our point of view, the importance of the feasible set and the efficient frontier is explained by the following result.

**Theorem 2.6.** *Let  $M_1$  and  $M_2$  be two non-degenerate Markowitz markets of dimension  $n$ . Let  $v_1$  and  $v_2$  be portfolios in  $M_1$  and  $M_2$  respectively, each of cost 1. Then there exists a Markowitz isomorphism of  $M_1$  and  $M_2$  sending  $v_1$  to  $v_2$  if and only if the efficient frontiers of  $M_1$  and  $M_2$  are equal and  $\phi(v_1) = \phi(v_2)$ .*

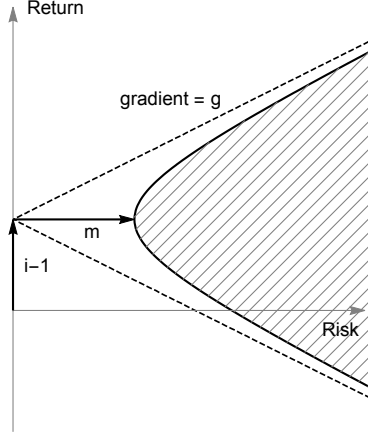


Figure 1: The efficient frontier (curved line) and the feasible set (shaded).

*Proof.* By assumption we are in either case (i) or case (ii) of Theorem 1.9. We are in case (ii) if and only if the efficient frontier is one arm of a hyperbola.

In case (ii), our explicit formula for the efficient frontier shows that  $m$ ,  $g$  and  $i$  can be recovered from its shape as shown in Figure 1.

After a rotation of the inner product space spanned by  $\{e_3, e_4, \dots, e_k\}$ , any portfolio in  $M_1$  of cost 1 can be written as  $(m, \lambda, \mu, 0, \dots, 0)$ . The  $\mu$  coefficient measures how far the image of  $v$  under  $\phi$  is to the right of the efficient frontier. The  $\lambda$  term identifies the point on the efficient frontier to the left of  $\phi(v)$ .

A similar argument can be applied in case (i).  $\square$

There is one feature of the market that is missed by risk-return diagrams, namely cost-free portfolios. These portfolios are not uninteresting. In our case (ii) the costless portfolio  $e_2$  provides one natural choice of mutual fund to use in the two mutual fund theorem. Adding multiples of this fund to your portfolio allows one to arbitrarily change the risk and return along the efficient frontier without affecting the cost. This fund is a particularly useful and easy to understand financial instrument.

Cost-free portfolios are also likely to be of great interest to rogue traders and fraudsters. They will want to know that arbitrarily large expected returns can be achieved in a Markowitz market at zero cost! Let us classify cost-free portfolios for their benefit. We omit the proof.

**Theorem 2.7.** Define  $\psi : V \rightarrow \mathbb{R}^2$  by  $\psi(v) = (\sqrt{r(v, v)}, p(v))$ . The image of the cost-free, risk-minimizing portfolios under  $\psi$  for the market  $\mathcal{M}_{k, m, g, i}^n$  with  $g > 0$  is the set  $(x, y) \in \mathbb{R}^2$  with  $x \geq 0$  and

$$y = \pm gx.$$

We call this set the efficient frontier for costless portfolios. The image of  $\psi$  is

either equal to the efficient frontier for costless portfolios or to the set of points on or to the right of the efficient frontier for costless portfolios.

There is an automorphism of the market mapping one costless portfolio to another if and only if they have the same image under  $\psi$ .

### 3 Dimension reduction of Markowitz markets

A corollary of our analysis is that there are no low-dimensional subspaces of a Markowitz model that can be naturally identified beyond those already identified by the mutual fund theorems.

**Theorem 3.1.** *For non-degenerate markets of dimension  $n$  containing no-valueless portfolios, any invariantly defined sub-manifold of the market has dimension less than or equal to 2 or greater than or equal to  $n - 2$ . If  $n > 4$  then the invariantly defined sub-manifolds of dimension less than or equal to 2 are all sub-manifolds of the set of risk-minimizing portfolios.*

*Proof.* Any sub-manifold of the market which is closed under the automorphism group of the market must consist of orbits of the automorphism group acting on  $V$ . As we have seen, excluding costless portfolios, these orbits consist of the pre-image of points of  $\phi$ . The pre-image of the efficient frontier has dimension less than or equal to 2. The pre-image of any other point in the feasible set is greater than or equal to  $n - 2$ . We use the map  $\psi$  to apply similar reasoning to the case of costless portfolios. It follows that invariant subspaces are of dimensions 0, 1, 2,  $n - 2$ ,  $n - 1$  or  $n$ . If  $n > 4$ ,  $n - 2 > 2$ . So in this case all low-dimensional invariant submanifolds are in the pre-image of the efficient frontiers. This implies they lie inside the set of risk-minimizing portfolios.  $\square$

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